LPV State-Feedback LMI Derivations

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1 Introduction

This document gives derivations of the *Linear Matrix Inequalities* (LMIs) that are used in order to perform state-feedback controller synthesis in LPVcore Toolbox using the `lpvsynsf` command.

2 Problem

Given a generalized plant

$$\xi x = A(p)x + B_{\mathrm{w}}(p)w + B_{\mathrm{u}}(p)u; \tag{1a}$$
$$z = C_{\mathrm{z}}(p)x + D_{\mathrm{zw}}(p)w + D_{\mathrm{zu}}(p)u; \tag{1b}$$

synthesize a state-feedback controller

$$u = K(p)x, \tag{2}$$

such that the closed-loop interconnection

$$\xi x = \overbrace{(A(p) + B_{\mathrm{u}}(p)K(p))}^{A_{\mathrm{cl}}(p)}\, x + \overbrace{B_{\mathrm{w}}(p)}^{B_{\mathrm{cl}}(p)}\, w; \tag{3a}$$
$$z = \underbrace{(C_{\mathrm{z}}(p) + D_{\mathrm{zu}}(p)K(p))}_{C_{\mathrm{cl}}(p)}\, x + \underbrace{D_{\mathrm{zw}}(p)}_{D_{\mathrm{cl}}(p)}\, w; \tag{3b}$$

satisfies a specific performance metric, such as minimal $\mathcal{L}_2$-gain or being passive.

3 Continuous-Time

3.1 $\mathcal{L}_2$-gain

From [1, Corollary 2.1], the closed-loop interconnection has a bounded $\mathcal{L}_2$-gain of γ if there exists a positive-definite matrix function M such that

$$\begin{bmatrix} A_{\mathrm{cl}}(p)^\top M(p) + (\star)^\top + \partial M(p,v) & M(p)B_{\mathrm{cl}}(p) & C_{\mathrm{cl}}(p)^\top \\ \star & -\gamma I & D_{\mathrm{cl}}(p)^\top \\ \star & \star & -\gamma I \end{bmatrix} \preceq 0. \tag{4}$$

Define $M^{-1}(p) = W(p)$, then

$$(\star)^\top \begin{bmatrix} A_{\mathrm{cl}}(p)^\top M(p) + (\star)^\top + \partial M(p,v) & M(p)B_{\mathrm{cl}}(p) & C_{\mathrm{cl}}(p)^\top \\ \star & -\gamma I & D_{\mathrm{cl}}(p)^\top \\ \star & \star & -\gamma I \end{bmatrix} \begin{bmatrix} M^{-1}(p) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \preceq 0, \tag{5}$$

which gives us

$$\begin{bmatrix} W(p)A_{\mathrm{cl}}(p)^\top + (\star)^\top - \partial W(p,v) & B_{\mathrm{cl}}(p) & W(p)C_{\mathrm{cl}}(p)^\top \\ \star & -\gamma I & D_{\mathrm{cl}}(p)^\top \\ \star & \star & -\gamma I \end{bmatrix} \preceq 0. \tag{6}$$

Note that $(\star)^\top(\partial M)M^{-1} = -\partial(M^{-1}) = -\partial W$ as $0 = \partial I = \partial(MM^{-1}) = (\partial M)M^{-1} + M(\partial(M^{-1}))$ [2].

Using (3), we have that

$$A_{\mathrm{cl}}(p)W(p) = A(p)W(p) + B_{\mathrm{u}}(p)K(p)W(p), \tag{7a}$$

$$C_{\mathrm{cl}}(p)W(p) = C_{\mathrm{z}}(p)W(p) + D_{\mathrm{zu}}(p)K(p)W(p). \tag{7b}$$

Let us define

$$F(p) := K(p)W(p). \tag{8}$$

Then, defining

$$\mathcal{A}_{\mathrm{cl}}(p,v) := A_{\mathrm{cl}}(p)W(p) - \frac{1}{2}\partial W(p,v) = A(p)W(p) + B_{\mathrm{u}}(p)F(p) - \frac{1}{2}\partial W(p,v), \tag{9a}$$

$$\mathcal{B}_{\mathrm{cl}}(p) := B_{\mathrm{cl}}(p) = B_{\mathrm{w}}(p), \tag{9b}$$

$$\mathcal{C}_{\mathrm{cl}}(p) := C_{\mathrm{cl}}(p)W(p) = C_{\mathrm{z}}(p)W(p) + D_{\mathrm{zu}}(p)F(p), \tag{9c}$$

$$\mathcal{D}_{\mathrm{cl}}(p) := D_{\mathrm{cl}}(p) = D_{\mathrm{zw}}(p), \tag{9d}$$

we have that (6) becomes

$$\begin{bmatrix} \mathcal{A}_{\mathrm{cl}}(p,v) + (\star)^\top & \mathcal{B}_{\mathrm{cl}}(p) & \mathcal{C}_{\mathrm{cl}}(p)^\top \\ \star & -\gamma I & \mathcal{D}_{\mathrm{cl}}(p)^\top \\ \star & \star & -\gamma I \end{bmatrix} \preceq 0. \tag{10}$$

After solving the set of LMIs defined by (10) for F and W, the controller K can be recovered by computing (based on (8))

$$K(p) = F(p)W(p)^{-1}. \tag{11}$$

3.2 Passivity

From [1, Corollary 2.2], the closed-loop interconnection is passive if there exists a positive-definite matrix function M such that

$$\begin{bmatrix} A_{\mathrm{cl}}(p)^\top M(p) + (\star)^\top + \partial M(p,v) & M(p)B_{\mathrm{cl}}(p) - C_{\mathrm{cl}}(p)^\top \\ \star & -D_{\mathrm{cl}}(p) + (\star)^\top \end{bmatrix} \preceq 0. \tag{12}$$

Again, define $M^{-1}(p) = W(p)$, then

$$(\star)^\top \begin{bmatrix} A_{\mathrm{cl}}(p)^\top M(p) + (\star)^\top + \partial M(p,v) & M(p)B_{\mathrm{cl}}(p) - C_{\mathrm{cl}}(p)^\top \\ \star & -D_{\mathrm{cl}}(p) + (\star)^\top \end{bmatrix} \begin{bmatrix} M(p)^{-1} & 0 \\ 0 & I \end{bmatrix} \preceq 0, \tag{13}$$

which gives us

$$\begin{bmatrix} W(p)A_{\mathrm{cl}}(p)^\top + (\star)^\top - \partial W(p,v) & B_{\mathrm{cl}}(p) - W(p)C_{\mathrm{cl}}(p)^\top \\ \star & -D_{\mathrm{cl}}(p) + (\star)^\top \end{bmatrix} \preceq 0. \tag{14}$$

Using (9), this becomes

$$\begin{bmatrix} \mathcal{A}_{\mathrm{cl}}(p,v) + (\star)^\top & \mathcal{B}_{\mathrm{cl}}(p) - \mathcal{C}_{\mathrm{cl}}(p)^\top \\ \star & -\mathcal{D}_{\mathrm{cl}}(p) + (\star)^\top \end{bmatrix} \preceq 0. \tag{15}$$

3.3 $\mathcal{L}_2$-$\mathcal{L}_\infty$-gain

From [1, Corollary 2.3], the closed-loop interconnection has a bounded $\mathcal{L}_2$-$\mathcal{L}_\infty$-gain of γ if there exists a positive-definite matrix function M such that

$$\begin{bmatrix} A_{\mathrm{cl}}(p)^\top M(p) + (\star)^\top + \partial M(p,v) & M(p)B_{\mathrm{cl}}(p) \\ \star & -\gamma I \end{bmatrix} \preceq 0, \qquad \begin{bmatrix} M(p) & C_{\mathrm{cl}}(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0. \tag{16}$$

Again, define $M^{-1}(p) = W(p)$, then

$$(\star)^\top \begin{bmatrix} A_{\mathrm{cl}}(p)^\top M(p) + (\star)^\top + \partial M(p,v) & M(p)B_{\mathrm{cl}}(p) \\ \star & -\gamma I \end{bmatrix} \begin{bmatrix} M(p)^{-1} & 0 \\ 0 & I \end{bmatrix} \preceq 0, \tag{17a}$$

$$(\star)^\top \begin{bmatrix} M(p) & C_{\mathrm{cl}}(p)^\top \\ \star & \gamma I \end{bmatrix} \begin{bmatrix} M(p)^{-1} & 0 \\ 0 & I \end{bmatrix} \preceq 0, \tag{17b}$$

which gives us

$$\begin{bmatrix} W(p)A_{\mathrm{cl}}(p)^\top + (\star)^\top - \partial W(p,v) & B_{\mathrm{cl}}(p) \\ \star & -\gamma I \end{bmatrix}, \qquad \begin{bmatrix} W(p) & W(p)C_{\mathrm{cl}}(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0. \tag{18}$$

Using (9), this becomes

$$\begin{bmatrix} \mathcal{A}_{\mathrm{cl}}(p,v) + (\star)^\top & \mathcal{B}_{\mathrm{cl}}(p) \\ \star & -\gamma I \end{bmatrix}, \qquad \begin{bmatrix} W(p) & \mathcal{C}_{\mathrm{cl}}(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0. \tag{19}$$

3.4 $\mathcal{L}_\infty$-gain

From [1, Corollary 2.4], the closed-loop interconnection has a bounded $\mathcal{L}_\infty$-gain of γ if there exists a positive-definite matrix function M and scalars $\alpha, \beta \geq 0$ such that

$$\begin{bmatrix} A_{\mathrm{cl}}(p)^\top M(p) + (\star)^\top + \beta M(p) + \partial M(p,v) & M(p)B_{\mathrm{cl}}(p) \\ \star & -\alpha I \end{bmatrix} \preceq 0, \qquad \begin{bmatrix} \beta M(p) & 0 & C_{\mathrm{cl}}(p)^\top \\ \star & (\gamma-\alpha)I & D_{\mathrm{cl}}(p)^\top \\ \star & \star & \gamma I \end{bmatrix} \succeq 0. \tag{20}$$

Again, define $M^{-1}(p) = W(p)$, then

$$(\star)^\top \begin{bmatrix} A_{\mathrm{cl}}(p)^\top M(p) + (\star)^\top + \beta M(p) + \partial M(p,v) & M(p)B_{\mathrm{cl}}(p) \\ \star & -\alpha I \end{bmatrix} \begin{bmatrix} M(p)^{-1} & 0 \\ 0 & I \end{bmatrix} \preceq 0, \tag{21a}$$

$$(\star)^\top \begin{bmatrix} \beta M(p) & 0 & C_{\mathrm{cl}}(p)^\top \\ \star & (\gamma-\alpha)I & D_{\mathrm{cl}}(p)^\top \\ \star & \star & \gamma I \end{bmatrix} \begin{bmatrix} M(p)^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \preceq 0, \tag{21b}$$

which gives us

$$\begin{bmatrix} W(p)A_{\mathrm{cl}}(p)^\top + (\star)^\top + \beta W(p) - \partial W(p,v) & B_{\mathrm{cl}}(p) \\ \star & -\alpha I \end{bmatrix}, \qquad \begin{bmatrix} \beta W(p) & 0 & W(p)C_{\mathrm{cl}}(p)^\top \\ \star & (\gamma-\alpha)I & D_{\mathrm{cl}}(p)^\top \\ \star & \star & \gamma I \end{bmatrix} \succeq 0. \tag{22}$$

Using (9), this becomes

$$\begin{bmatrix} \mathcal{A}_{\mathrm{cl}}(p,v) + (\star)^\top + \beta W(p) & \mathcal{B}_{\mathrm{cl}}(p) \\ \star & -\alpha I \end{bmatrix}, \qquad \begin{bmatrix} \beta W(p) & 0 & \mathcal{C}_{\mathrm{cl}}(p)^\top \\ \star & (\gamma-\alpha)I & \mathcal{D}_{\mathrm{cl}}(p)^\top \\ \star & \star & \gamma I \end{bmatrix} \succeq 0. \tag{23}$$

4 Discrete-Time

4.1 ℓ_2-gain

Following the derivations in [1, A.3.1 - Discrete Time], it holds (see [1, (A.149)]) that the closed-loop interconnection has a bounded ℓ_2-gain of γ (corresponding to $(Q,S,R) = (\gamma I, 0, -\gamma^{-1} I)$) if there exists a positive-definite matrix function M and matrix G such that

$$\begin{bmatrix} M(p+v) & A_{\mathrm{cl}}(p)G & B_{\mathrm{cl}}(p) \\ \star & G + (\star)^\top - M(p) & 0 \\ \star & \star & 0 \end{bmatrix} + (\star)^\top \begin{bmatrix} \gamma I & 0 \\ \star & \gamma^{-1} I \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ 0 & C_{\mathrm{cl}}(p)G & D_{\mathrm{cl}}(p) \end{bmatrix} \succeq 0, \tag{24}$$

which is equivalent to

$$\begin{bmatrix} M(p+v) & A_{\mathrm{cl}}(p)G & B_{\mathrm{cl}}(p) \\ \star & G + (\star)^\top - M(p) & 0 \\ \star & \star & \gamma I \end{bmatrix} + (\star)^\top (\gamma I)^{-1} \begin{bmatrix} 0 & C_{\mathrm{cl}}(p)G & D_{\mathrm{cl}}(p) \end{bmatrix} \succeq 0, \tag{25}$$

and using a Schur complement to

$$\begin{bmatrix} M(p+v) & A_{\mathrm{cl}}(p)G & B_{\mathrm{cl}}(p) & 0 \\ \star & G + (\star)^\top - M(p) & 0 & G^\top C_{\mathrm{cl}}(p)^\top \\ \star & \star & \gamma I & D_{\mathrm{cl}}(p)^\top \\ \star & \star & \star & \gamma I \end{bmatrix} \succeq 0. \tag{26}$$

Using (3), we have that

$$A_{\mathrm{cl}}(p)G = A(p)G + B_{\mathrm{u}}(p)K(p)G, \tag{27a}$$

$$C_{\mathrm{cl}}(p)W(p) = C_{\mathrm{z}}(p)G + D_{\mathrm{zu}}(p)K(p)G. \tag{27b}$$

Let us define

$$F(p) := K(p)G. \tag{28}$$

Then, defining

$$\mathcal{A}_{\mathrm{cl}}(p) := A_{\mathrm{cl}}(p)G = A(p)G + B_{\mathrm{u}}(p)F(p), \tag{29a}$$

$$\mathcal{B}_{\mathrm{cl}}(p) := B_{\mathrm{cl}}(p) = B_{\mathrm{w}}(p), \tag{29b}$$

$$\mathcal{C}_{\mathrm{cl}}(p) := C_{\mathrm{cl}}(p)G = C_{\mathrm{z}}(p)G(p) + D_{\mathrm{zu}}(p)F(p), \tag{29c}$$

$$\mathcal{D}_{\mathrm{cl}}(p) := D_{\mathrm{cl}}(p) = D_{\mathrm{zw}}(p), \tag{29d}$$

we have that (26) becomes

$$\begin{bmatrix} M(p+v) & \mathcal{A}_{\mathrm{cl}}(p) & \mathcal{B}_{\mathrm{cl}}(p) & 0 \\ \star & G + (\star)^\top - M(p) & 0 & \mathcal{C}_{\mathrm{cl}}(p)^\top \\ \star & \star & \gamma I & \mathcal{D}_{\mathrm{cl}}(p)^\top \\ \star & \star & \star & \gamma I \end{bmatrix} \succeq 0. \tag{30}$$

After solving the set of LMIs defined by (10) for F and W, the controller K can be recovered by computing (based on (28))

$$K(p) = F(p)G^{-1}. \tag{31}$$

4.2 Passivity

Following the derivations in [1, A.3.1 - Discrete Time], it holds (see [1, (A.149)]) that the closed-loop interconnection is passive (corresponding to $(Q, S, R) = (0, I, 0)$) if there exists a positive-definite matrix function M and matrix G such that

$$\begin{bmatrix} M(p+v) & A_{\mathrm{cl}}(p)G & B_{\mathrm{cl}}(p) \\ \star & G+(\star)^\top - M(p) & 0 \\ \star & \star & 0 \end{bmatrix} + (\star)^\top \begin{bmatrix} 0 & I \\ \star & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ 0 & C_{\mathrm{cl}}(p)G & D_{\mathrm{cl}}(p) \end{bmatrix} \succeq 0, \tag{32}$$

which is equivalent to

$$\begin{bmatrix} M(p+v) & A_{\mathrm{cl}}(p)G & B_{\mathrm{cl}}(p) \\ \star & G+(\star)^\top - M(p) & G^\top C_{\mathrm{cl}}(p)^\top \\ \star & \star & D_{\mathrm{cl}}(p) + (\star)^\top \end{bmatrix} \succeq 0. \tag{33}$$

Using (29), this becomes

$$\begin{bmatrix} M(p+v) & \mathcal{A}_{\mathrm{cl}}(p) & \mathcal{B}_{\mathrm{cl}}(p) \\ \star & G+(\star)^\top - M(p) & \mathcal{C}_{\mathrm{cl}}(p)^\top \\ \star & \star & \mathcal{D}_{\mathrm{cl}}(p) + (\star)^\top \end{bmatrix} \succeq 0. \tag{34}$$

4.3 ℓ_2-ℓ_∞-gain

Following the derivations in [1, A.3.1 - Discrete Time], it holds (see [1, (A.149)]) that the closed-loop interconnection has a bounded ℓ_2-ℓ_∞-gain of γ (corresponding to $(Q, S, R) = (\gamma I, 0, 0)$) if there exists a positive-definite matrix function M and matrix G such that

$$\begin{bmatrix} M(p+v) & A_{\mathrm{cl}}(p)G & B_{\mathrm{cl}}(p) \\ \star & G+(\star)^\top - M(p) & 0 \\ \star & \star & 0 \end{bmatrix} + (\star)^\top \begin{bmatrix} \gamma I & 0 \\ \star & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ 0 & C_{\mathrm{cl}}(p)G & D_{\mathrm{cl}}(p) \end{bmatrix} \succeq 0, \tag{35a}$$

and from [1, (A.202)]

$$\begin{bmatrix} G+(\star)^\top - M(p) & G^\top C_{\mathrm{cl}}(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0, \tag{35b}$$

which is equivalent to

$$\begin{bmatrix} M(p+v) & A_{\mathrm{cl}}(p)G & B_{\mathrm{cl}}(p) \\ \star & G+(\star)^\top - M(p) & 0 \\ \star & \star & \gamma I \end{bmatrix} \succeq 0, \qquad \begin{bmatrix} G+(\star)^\top - M(p) & G^\top C_{\mathrm{cl}}(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0. \tag{36}$$

Using (29), this becomes

$$\begin{bmatrix} M(p+v) & \mathcal{A}_{\mathrm{cl}}(p) & \mathcal{B}_{\mathrm{cl}}(p) \\ \star & G+(\star)^\top - M(p) & 0 \\ \star & \star & \gamma I \end{bmatrix} \succeq 0, \qquad \begin{bmatrix} G+(\star)^\top - M(p) & \mathcal{C}_{\mathrm{cl}}(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0. \tag{37}$$

4.4 ℓ_∞-gain

Based on [1, (A.215)], it holds that the closed-loop interconnection has a bounded ℓ_∞-gain of γ if there exists a positive-definite matrix function M and matrix G such that

$$\begin{bmatrix} M(p+v) & A_{\mathrm{cl}}(p)G & B_{\mathrm{cl}}(p) \\ \star & (1-\beta)(G+(\star)^\top - M(p)) & 0 \\ \star & \star & \alpha I \end{bmatrix} \succeq 0, \tag{38a}$$

and from [1, (A.220)]

$$\begin{bmatrix} \beta(G+(\star)^\top - M(p)) & 0 & G^\top C_{\mathrm{cl}}(p)^\top \\ \star & (\gamma-\alpha)I & D_{\mathrm{cl}}(p)^\top \\ \star & \star & \gamma I \end{bmatrix} \succeq 0. \tag{38b}$$

Using (29), this becomes

$$\begin{bmatrix} M(p+v) & \mathcal{A}_{\mathrm{cl}}(p) & \mathcal{B}_{\mathrm{cl}}(p) \\ \star & (1-\beta)(G+(\star)^\top - M(p)) & 0 \\ \star & \star & \alpha I \end{bmatrix} \succeq 0, \tag{39a}$$

and

$$\begin{bmatrix} \beta(G+(\star)^\top - M(p)) & 0 & \mathcal{C}_{\mathrm{cl}}(p)^\top \\ \star & (\gamma-\alpha)I & \mathcal{D}_{\mathrm{cl}}(p)^\top \\ \star & \star & \gamma I \end{bmatrix} \succeq 0. \tag{39b}$$

Note that the LMIs in (10), (15), (19), (23), (30), (34), (37), and (39) have the same form as the output-feedback controller synthesis LMIs in Corrolaries 2.5 - 2.8 in [1].

References

[1] P. J. W. Koelewijn, *Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees*. PhD thesis, Eindhoven University of Technology, 2023.

[2] A.Γ., "Derivative of the inverse of a matrix." Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/1471836 (version: 2015-10-09).